Abstract

A Gaussian filter using the Hermite orthonormal series of functions is developed. The filter is compared with a similar filter using the Hermite-Rodriguez series on Doppler radar signals. The results indicate that a more compact filter can be achieved with the Hermite series compared to the Hermite-Rodriguez series.

1 Introduction

Figure 1 explains the operation of a classical filter, sometimes referred to as a moving average filter. The filter integrates only the noisy function, \( f(\tau) \), enclosed within the window function, \( w(\tau - \tau) \), at the location \( t \). An average value, \( \hat{f}(t) \), of the noisy function at \( t \) is obtained. This average value is the correlation of the noisy function with the window function. With a Gaussian window function this is:

\[
\hat{f}(t) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) \exp \left( -\frac{(\tau - \tau)^2}{2\sigma^2} \right) d\tau \tag{1}
\]

where

\[
w(t) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{t^2}{2\sigma^2} \right) \tag{2}
\]

This integration can be approximated by a discrete summation in digital applications, or as a Monte Carlo integration in statistics, which is referred to as Kernel regression. In this paper we investigate the computation of this correlation using an orthogonal series. A large number of well-known orthogonal series occur including Fourier, Legendre, and the Tchebychev series. While these are useful in certain applications, in this work we choose the Hermite and Hermite-Rodriguez orthogonal series, which are based on the Gaussian function because this simplifies the calculation of the correlation.

The layout of this paper is as follows. Section 2 explains how the correlation is calculated using orthogonal functions. Section 3 describes the spatial and frequency bandwidth of the orthogonal series, which is important in applying the series to real life problems. Section 4 applies the series to demodulate Doppler radar signals.

2 Orthogonal filters

2.1 Hermite series

Consider the approximation of the noisy function by a Hermite series. The series approximates the function by a finite expansion of Hermite functions on an interval \([ -\infty, +\infty ) \):

\[
f(t) \equiv f_N(t) = \sum_{n=0}^{N} a_n h_n(t) \quad \{-\infty \leq t \leq \infty \} \tag{3}
\]

where \( \{a_n\} \) are a set of suitably chosen weights and \( h_n(t) \) are the Hermite activation functions on the interval \([ -\infty, +\infty ) \) which satisfy the orthonormal condition

\[
\int_{-\infty}^{\infty} h_m(t) h_n(t) dt = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \tag{4}
\]

The first few Hermite functions in this series are

\[
h_0(t) = \frac{\exp(-t^2/2)}{\sqrt{\pi}} \tag{5}
\]

\[
h_1(t) = \frac{2t \exp(-t^2/2)}{\sqrt{\pi} \cdot 2^{1/2}} \tag{6}
\]

The remainder may be determined from the recurrence relation

\[
h_{n+1}(t) = t \sqrt{\frac{2}{n+1}} h_n(t) - \sqrt{n \cdot \frac{n}{n+1}} h_{n-1}(t) \tag{7}
\]
Since the fundamental, \( h_n(t) \), is the Gaussian function it can take the place of the filter function. The correlation integration of equation (1) is then approximately equal to the correlation of the Hermite series with the fundamental Hermite function.

\[
\hat{f}(t) \equiv \hat{f}_N(t) = \int_r^\infty \left[ \sum_{n=0}^{N} a_n h_n(\tau) \right] h_0(t-\tau) d\tau \quad (8)
\]

In order to evaluate this expression, the correlation between the Hermite functions of different order, is required. This correlation is given by

\[
\int_{-\infty}^{\infty} h_n(\tau) h_m(t-\tau) d\tau = \frac{1}{2} l_m^n (t^2/2) \quad (m \leq n) \quad (9)
\]

where \( l_m^n(t) \) is a normalized associated Laguerre function. Using this result in equation (8) we obtain

\[
\hat{f}_N(t) = \frac{1}{2} \sum_{n=0}^{N} a_n l_n^0 (t^2/2) \quad (10)
\]

Note that only those associated Laguerre functions of subscript {\( m = 0 \)} contribute, thus greatly simplifying the result. The associated Laguerre functions of subscript are

\[
l_n^0(t) = \frac{t^{n/2}e^{-t/2}}{\sqrt{n!}} \quad (11)
\]

The usefulness of this expansion is that by fitting a Hermite series to the input function, one also immediately obtains the weights of the Laguerre series, \( \{a_n\} \), which is the correlation of the input with a Gaussian function.

### 2.2 Hermite Rodriguez Functions

Hermite-Rodriguez functions are similar to the Hermite functions except that a Gaussian window modulates their amplitude. They are defined as

\[
hr_n(t) = (\pi)^{1/4} h_n(t) e^{-t^2/2} \quad (12)
\]

where \( h_n(t) \) is an orthonormal Hermite function.

The fundamental Hermite-Rodriguez function is also a Gaussian function but of different width to the fundamental Hermite function. The fundamental is

\[
hr_0(t) = \exp(-t^2) \quad (14)
\]

The others may be determined using the recurrence relation (equation (7)) for the Hermite functions and multiplying by the Gaussian function.

Like the Hermite series, a simple expression also occurs for the correlation of the Hermite-Rodriguez functions. The correlation of two Hermite-Rodriguez functions is given by

\[
hr_n(t) * hr_m(t) = \sqrt{\frac{(n+m)!}{n!m!}} hr_{n+m}(t/\sqrt{2}) \quad (15)
\]

Note that the scale is reduced and the order of the Hermite-Rodriguez function increased by the correlation operation.

Using equation (15), the derivation of a Gaussian filter with the Hermite-Rodriguez functions is similar to the filter derived using the Hermite series.

### 3 Method

#### 3.1 Signal duration and bandwidth

Application requires the duration and bandwidth of the orthogonal series to be matched to the signal being modeled.

The Hermite series behaves as a window in the time domain. Outside this window the Hermite functions decay exponentially, limiting the effective range over which a function may be approximated to within the window. The width of the Hermite series window is equal to the duration of the largest order Hermite function, \( h_N(t) \), occurring in the series. The useful range of application of the Hermite series interpolation is then

\[
h \leq \sqrt{2N + 1} \quad (16)
\]

Where the right hand side, equal to the duration of the Hermite function of order \( N \), may be determined via the Quantum mechanic solution of the Harmonic oscillator as the location where the oscillator energy becomes negative.

The Fourier transform of a Hermite function is

\[
F \{h_n(t)\} \rightarrow j^n h_n(\omega) \quad (17)
\]

In view of this isomorphic Fourier transform, a similar windowing effect occurs in the complex frequency domain. The useful bandwidth is
\[ |a| \leq \sqrt{2N + 1} \quad (18) \]

Together, the bandwidth and window width of equation (20) and (21) define the size, \( N \), of the neural series required to approximate a function.

Unlike the Hermite series, which increases in duration with the order \( \{N\} \) of the function, the Hermite-Rodriguez series is independent of \( N \). Instead it is limited in duration by the Gaussian amplitude modulation function to the range\(^5\)
\[ 3t \leq (19) \]

The Fourier transform of the Hermite-Rodriguez function is an associated Laguerre function\(\)\(\)\(\)
\[ F\{hr_n(t)\} = (-j)^n t^n (\alpha^2/2) \quad (20) \]

From this Fourier transform it can be shown that the bandwidth in the frequency domain increases with the order \( \{N\} \) of the function according to
\[ |\omega| \leq 3 \quad (19) \]

The scaled Hermite-Rodriguez series is obtained by introducing the variable
\[ t \rightarrow t/\alpha \quad (22) \]

Scaling changes the duration and bandwidth of the Hermite-Rodriguez series to, respectively,
\[ |\omega| \leq 3\alpha \quad (23) \]
and
\[ |a| \leq \sqrt{2N + 1}/\alpha \quad (24) \]

Using scaled Hermite-Rodriguez functions\(^5\), the correlation with the Gaussian function is
\[ hr_n(t/\alpha) * hr_n(t/\beta) = \left(\frac{\alpha}{\gamma}\right)^n hr_n(t/\gamma) \quad (25) \]

where \( \alpha \) and \( \beta \) are the scaling factors and \( \gamma^2 = \alpha^2 + \beta^2 \).

### 3.3 Optimising the weights of the orthogonal series

The weights of the Hermite and Hermite-Rodriguez series were both obtained using the same method, which is described in this section for the Hermite series. For a continuous function defined on \( \{-\infty, \infty\} \), the weights of the Hermite series are optimum\(^7\) with respect to the mean square error (equation (3)) when
\[ A_n = \int_{-\infty}^{\infty} a(t) h_n(t) \, dt \quad (26) \]

We use a simple summation, similar to Euler integration, given by
\[ A_n = \sum_{i=0}^{\infty} a(\Delta t) h_n(\Delta t) \quad (27) \]

where \( \Delta t \) is the integration step size which was fixed to the sampling rate. A feature of this type of integration is that it is also suitable for randomly sampled data. For random data, this type of numerical integration generalizes to Monte-Carlo integration.

Numerical integration is only an approximation to the analytical continuous integration. In addition, the data most often encountered in practice is discrete, often corrupted with noise. To cope with these situations the gradient descent algorithm\(^8\) was applied after the weights had been estimated with the integration. Gradient descent reduces the mean square error between the Hermite series approximation and the discrete data by successive iterations of the following algorithm
\[ A_{k+1} = A_k + \mu \left( a(t) - \sum_{n=0}^{N} A_n h_n(t) h_n(t) \right) \quad (28) \]

where \( \mu \) is the feedback constant chosen in the range 0.0 to 1.0.

### 4 Application to the demodulation of Doppler radar signals
The objective of this section is to compare the performance of the Hermite with the Hermite-Rodriguez series on a practical signal processing problem.

Figure 2(a) shows the signal received from the detector of a Doppler radar system. The carrier frequency, which is proportional to the velocity of the target, is removed by filtering to obtain the target range. The frequency spectrum of the Doppler signal, from which the velocity may be determined, is given by the new series formed by replacing the Hermite functions in the Hermite series interpolation of the Doppler signal by their Fourier transform (equation (9)). Removal of the carrier frequency to obtain the range of the target is achieved by taking the correlation of the Hermite series with the Gaussian function.

The following mathematical model of a Doppler signal was investigated

\[ a(t) = (1 + \cos(2\pi f t)) e^{-t^2/2\sigma^2} \quad (26) \]

where the frequency, \( f = 1\text{Hz} \), and the Gaussian width, \( \sigma = 3.0 \). Simulated random noise, with a uniform probability density, of varying strength was added. A Hermite series of 50 elements has a width of 10.0 seconds and a bandwidth of 1.5Hz, which is sufficient to accurately interpolate the Doppler signal. Figure 2(a), 2(b), and 2(c) show the signal, Gaussian filtered output and frequency spectrum respectively from the Hermite series interpolation containing 50 elements. The root mean square error (RMSE) of the Hermite series interpolation of the Doppler signal versus the number of Hermite functions in the series is shown in figure 3(a).

For the purpose of comparison, the scaling parameters were chosen so that the Hermite-Rodriguez series matches the Hermite series. With \( \beta = \sqrt{2} \), the order \( \{n=0\} \), Hermite-Rodriguez function is matched exactly to the order \( \{n=0\} \), Hermite function, that is \( h_n(t/2) = h_n(t) \).

Similarly, with \( \alpha = 2.85\sqrt{2} \), the Hermite-Rodriguez series is matched in duration to the Hermite series. Figure 3(b), shows the root mean square error of the Hermite-Rodriguez interpolation as a function of the number of elements of the series. More than 400 Hermite-Rodriguez functions are needed before the error drops to a sufficiently small value. The reason for this is because the frequency response of the scaled Hermite-Rodriguez series is considerably smaller than a Hermite series of the same duration. Figure 4(a) and 4(b) shows the frequency response of the Hermite series and Hermite-Rodriguez series on a unit amplitude cosine with 50 and 400 Hermite-Rodriguez functions respectively. The poor frequency response is unavoidable because a sufficiently large \( \alpha \) must be chosen to ensure the series has sufficient duration to interpolate the Doppler signal.

Figure 5 shows the output signal to noise ratio of the Hermite series, \( N = 50 \), and the Hermite-Rodriguez series, \( N = 400 \), versus the input signal to noise ratio. Nearly identical signal to noise ratios are achieved, which is to be expected since an identical Gaussian correlation function was used, but the Hermite series is considerably more efficient since it achieves the same result with much fewer terms in the series. In view of their Gaussian window, the Hermite-Rodriguez series may prove more suitable for correctly analyzing the frequency spectrum of signals subject to glitches.

Conclusion

A Gaussian filter using Hermite functions was developed. A comparison with a similar filter using the Hermite-Rodriguez series favours the Hermite series because it has a greater bandwidth for a series with the same number of elements. A comparison of the Hermite Gaussian filter with a Gaussian filter using Fourier functions is under investigation.

References


Figure 1 Operation of a moving average filter

Figure 2(a) Doppler radar signal

Figure 2(b) Hermite filtered Doppler radar signal
Figure 2(c) Power spectrum of Doppler radar signal by Hermite series.

Figure 3(a) Root mean square error versus number of Hermite Functions.

Figure 3(b) Root mean square error versus number of Hermite-Rodriguez functions.

Figure 4(a) Frequency response of Hermite and Hermite-Rodriguez series with 50 elements.

Figure 4(b) Frequency response of Hermite (50 elements) and Hermite-Rodriguez series (400 elements).

Figure 5 Comparative signal to noise ratios of Hermite and Hermite-Rodriguez Doppler radar demodulator.